英

令和４年度 秋季募集
(令和５年４月入学)

東北大学大学院工学研究科
量子エネルギー工学専攻入学試験

試験問題冊子

数学Ａ　MATHEMATICS A

令和 4 年 8 月 30 日(火)
Tuesday, August 30, 2022　　10:00 － 11:30

Notice

1. Do not open this examination booklet until instructed to do so.

2. An examination booklet, answer sheets, draft sheets are provided. Put your examinee number on each of the answer sheets and the draft sheets.

3. Answer all problems.

4. At the end of the examination, double-check your examinee number and the problem numbers on the answer sheets. Put your answer sheets in numerical order on your draft sheet, place them beside the test booklet, and wait for collection by an examiner. Do not leave your seat before instructed to do so by the examiner.

数 学 A　MATHEMATICS A

1. Calculate the following integrals.

(1) $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}$

(2) $\displaystyle\iint_D \frac{x^3}{y^2}\, dx\, dy$,　$D = \{(x,y) \mid x^4 \le y \le x,\ \frac{1}{2} \le x \le 1\}$

(3) $\displaystyle\iint_D \exp(\cos(x))\, dx\, dy$,　$D = \{(x,y) \mid \sin^{-1} y \le x \le \frac{\pi}{2},\ 0 \le y \le 1\}$

2. As shown in Fig. 1, $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are the fundamental vectors of the three-dimensional Cartesian coordinate system (x, y, z), and $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$, and $\boldsymbol{e}_\phi$ are those of the three-dimensional polar coordinate system (r, θ, ϕ). The length of the fundamental vectors is 1. In the three-dimensional Cartesian coordinate system, the vector field $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = x\sqrt{x^2+y^2+z^2}\,\boldsymbol{i} + y\sqrt{x^2+y^2+z^2}\,\boldsymbol{j} + z\sqrt{x^2+y^2+z^2}\,\boldsymbol{k} \ .$$

Here, S is defined as the surface of the following region V,

$$V = \{(x,y,z) \mid a^2 \le x^2+y^2+z^2 \le b^2,\ z \ge 0\},$$

where a and b are real numbers satisfying $0 < a < b$. Solve the following problems.

(1) Obtain $\nabla \cdot \boldsymbol{A}$ and $\nabla \times \boldsymbol{A}$ in the three-dimensional Cartesian coordinate system.

(2) Express x, y, and z using r, θ, and ϕ.

(3) Obtain the Jacobian while converting three-dimensional Cartesian coordinate system to three-dimensional polar system.

(4) Evaluate the integral $\int_S \boldsymbol{A} \cdot \boldsymbol{n}\, dS$, where $\boldsymbol{n}$ is the outward unit normal vector of S.

(5) Express $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$, and $\boldsymbol{e}_\phi$ with $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, θ, and ϕ.

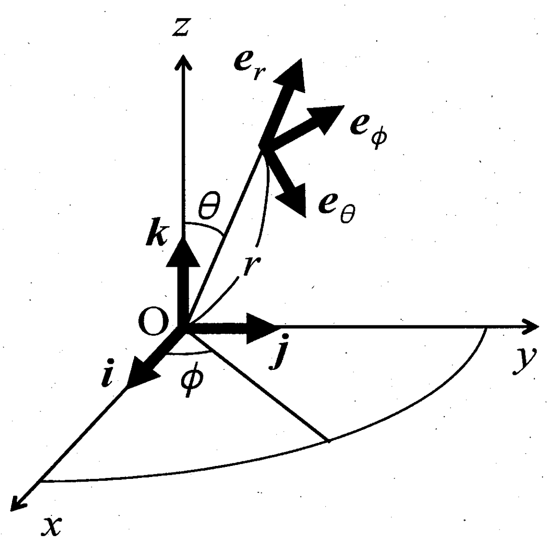

Fig. 1

3. Consider the following function,

$$f(x,y) = x^2 + y^2 - 2\alpha xy$$

where α is a positive constant of $0 < \alpha < 2$. Solve the folloing problems.

(1) Find a symmetric matrix A when $f(x,y)$ is rewritten as follows,

$$f(x,y) = \begin{pmatrix} x & y \end{pmatrix} A \begin{pmatrix} x \\ y \end{pmatrix} .$$

(2) Find two different eigenvalues λ_1 and λ_2 $(\lambda_1 > \lambda_2)$, and corresponding unit eigenvectors, $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$, of the matrix A obtained in problem (1), which satisfy the following relation,

$$\det(\boldsymbol{p}_1 \;\; \boldsymbol{p}_2) = 1 .$$

(3) Using the result obtained in problem (2), show the following relation,

$$f(x,y) = \begin{pmatrix} X & Y \end{pmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} .$$

(4) Using the relation in problem (3), explain how the shape of curve, $f(x,y) = 1$, changes according to the value of α.